**Journal of Research in Physics** Vol. 34, No. 1, 23 - 32 (2010)

# Ground state wave functions of superconductivity on honeycomb lattice

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Received: *November 22, 2010*

**Abstract**

We derive ground state wave functions of superconducting instabilities on the honeycomb lattice induced by nearest-neighbor attractive interactions. They reflect the Dirac nature of electrons in the low-energy limit. For the order parameter that is the same irrespective of the direction to any of the nearest neighbors we find weak pairing (slowly decaying) behavior in the orbital part of the Cooper pair with no angular dependence. At the neutrality point, in the spin-singlet case, we recover a strong pairing behavior.

## 1. Introduction

The advent of graphene [1] opened a door for exploration of new phenomena in two-dimensional Dirac-like condensed matter systems. One of the intriguing questions is concerned with superconducting correlations of electrons on the honeycomb lattice system. Superconductivity has been induced in short graphene samples through proximity effect with superconducting contacts [2]. This indicates that Cooper pairs can propagate coherently in th graphene. In principle, superconductivity on the graphene honeycomb lattice can be induced by short-range attractive interactions and explorations of allowed possibilities were given in [3, 4, 5]. Among the most interesting is the so-called $p + ip$ superconducting instability introduced in [3]. It is supported by the most natural nearest-neighbor attractive interaction and have distinct features of the Dirac electrons. Later, it has been shown [5], by a restricted (low-energy) analysis, that this state may be less energetically favorable with respect to Kekule-like order parameter arrangements. Nevertheless, the $p + ip$ instability seems, though an exotic state, a very attractive possibility because of its underlying symmetry of the order parameter, the same as for Pfaffian quantum Hall state [6] or $p+ip$ spinless superconductor [7]. The latter systems support non-Abelian statistics, which is at the heart of the idea of the topological computing [8]. There is an important difference between these states and the proposed graphene state. The superconducting instability in graphene does not break time-reversal symmetry and those systems do. Due to the valley degeneracy we effectively have two $(p \pm ip)$ order parameters and this requires additional understanding of intertwined correlations and underlying symmetries. One way, just as in the Pfaffian state [7], is to look for the ground state wave function and recognize the structures and symmetries.

In this paper we derive an effective (long-distance) expression for the ground state wave function of the $p + ip$ spin-singlet instability described in Ref. 3 and display the pertinent symmetries in this case. Also, a spinless case will be discussed. We will use the BCS mean-field formalism. In the following section we will set up the BCS formalism, solve the Bogoliubov - de Gennes (BdG) equations and find the expression for the ground state wave functions. The last section is devoted to conclusions.

## 2. Superconductivity on honeycomb lattice and its ground states

The Hamiltonian for free electrons on the honeycomb lattice is

$$H_0 = -t \sum_{\langle ij \rangle} \sum_{\sigma=\uparrow,\downarrow} (a^\dagger_{i,\sigma} b_{j,\sigma} + h.c.) - \mu \sum_i \hat{n}_i, \tag{1}$$

where $t$ is the hopping energy between nearest neighbor C (carbon) atoms, $a_{i,\sigma}(a^\dagger_{i,\sigma})$ is the on-site annihilation (creation) operator for electrons in the sublattice A with spin $\sigma =\uparrow,\downarrow$, and $b_{i,\sigma}(b^\dagger_{i,\sigma})$ for sublattice B, $\hat{n}_i$ is the on-site number operator, and $\mu$ is the graphene chemical potential. We use units such that $\hbar = 1$. Diagonalization of Eq.(1) leads to a spectrum given by: $\epsilon_{\vec{k}} = \pm t|S(\vec{k})|$, where $\vec{k}$ is the two-dimensional momentum, and $S(\vec{k}) = \sum_{\vec{\delta}} \exp\{i\vec{k}\vec{\delta}\}$ with $\vec{\delta}$'s defined as $\delta_1 = a(0, 1/\sqrt{3})$, $\delta_2 = a/2(1, -1/\sqrt{3})$, and $\delta_3 = a/2(-1, -1/\sqrt{3})$, and $a = \sqrt{3}\ a_{cc}$, $a_{cc}$ is the distance between $C$ atoms and $a$ is the next to nearest neighbor distance. At the corners of the hexagonal Brillouin zone, $\vec{K}_\pm = (2\pi)/a(2/3, 0)$, we have $S(\vec{K}_\pm + \vec{k}) \approx \mp a\sqrt{3}/2(k_x \mp ik_y)$, and the band has the shape of a Dirac cone: $\epsilon(\vec{K}_\pm + \vec{k}) = \pm v_F|\vec{k}|$, where $v_F = (\sqrt{3}at)/2$ is the Fermi-Dirac velocity.

For the sake of simplicity we will consider only nearest-neighbor attractive interactions among electrons. The on-site repulsive interactions can be introduced and will not change our conclusions. Therefore the complete Hamiltonian will include nearest-neighbor interactions as follows,

$$H_I = g \sum_{\langle ij \rangle} \sum_{\sigma,\sigma'} a^\dagger_{i,\sigma} a_{i,\sigma} b^\dagger_{j,\sigma'} b_{j,\sigma'}, \tag{2}$$

where $g < 0$. We will assume the spin-singlet pairing among nearest-neighbors and apply the BCS ansatz with $\Delta_{ij} = \langle a_{i,\downarrow} b_{j,\uparrow} - a_{i,\uparrow} b_{j,\downarrow} \rangle$, the superconducting order parameter. Furthermore we assume one and the same $\Delta_{ij} = \Delta$ for all nearest neighbors, which, due to global gauge ($U(1)$) transformations on $a$'s and $b$'s, can be chosen real and positive. The interaction part, $H_I$, becomes

$$\tilde{H}_{BCS} = \{g \sum_{\langle ij \rangle} \Delta(a^\dagger_{i,\uparrow} b^\dagger_{j,\downarrow} - a^\dagger_{i,\downarrow} b^\dagger_{j,\uparrow}) + h.c.\} - 3g|\Delta|^2. \tag{3}$$

The order parameter in the momentum space is

$$\Delta_{\vec{k}} = \sum_{\langle ij \rangle} \Delta \exp\{i\vec{k}(\vec{i} - \vec{j})\} = \Delta \sum_{\vec{\delta}} \exp\{i\vec{k}\vec{\delta}\} = \Delta S(\vec{k}). \tag{4}$$

Therefore near $K$ points $\Delta_{\vec{K}_\pm + \vec{k}} \sim \mp(k_x \mp ik_y)$, which then describes two $p$-wave like superconducting order parameters in a low effective description. The complete BCS Hamiltonian can be now cast in the following form in the momentum space,

$$H_{BCS} = \sum_{\vec{k}} \phi^\dagger_{\vec{k}} M_{\vec{k}} \phi_{\vec{k}}, \tag{5}$$

where

$$\phi^\dagger_{\vec{k}} = (a^\dagger_{\vec{k}\uparrow}, b^\dagger_{\vec{k}\uparrow} a_{-\vec{k}\downarrow}, b_{-\vec{k}\downarrow}), \tag{6}$$

with defined $a_{\vec{k}\sigma} = \sum_i a_{i\sigma} \exp\{i\vec{k}\,\vec{i}\}$ and $b_{\vec{k}\sigma} = \sum_i b_{i\sigma} \exp\{i\vec{k}\,\vec{i}\}$, and, with $g\Delta \equiv \Delta$ for short,

$$M_{\vec{k}} = \begin{bmatrix} -\mu & -tS(\vec{k}) & 0 & \Delta S(\vec{k}) \\ -tS^*(\vec{k}) & -\mu & \Delta S(-\vec{k}) & 0 \\ 0 & \Delta S^*(-\vec{k}) & \mu & tS(\vec{k}) \\ \Delta S^*(\vec{k}) & 0 & tS^*(\vec{k}) & \mu \end{bmatrix}.$$

We look for the solution in the form of a diagonalized Bogoliubov BCS Hamiltonian,

$$H_{BCS} = \sum_{\vec{k},\gamma=\pm} \omega^\alpha_{\vec{k},\gamma} \alpha^\dagger_{\vec{k},\gamma} \alpha_{\vec{k},\gamma} + \sum_{\vec{k},\gamma=\pm} \omega^\beta_{\vec{k},\gamma} \beta^\dagger_{\vec{k},\gamma} \beta_{\vec{k},\gamma} + E_0, \tag{7}$$

where $\alpha_{\vec{k},\gamma}$ and $\beta_{\vec{k},\gamma}$, $\gamma = \pm$ are new quasiparticles at momentum $\vec{k}$. For the dispersions we have:

$$\omega^\alpha_{\vec{k},\gamma} = \gamma\omega^\alpha_{\vec{k}} \quad \text{and} \quad \omega^\beta_{\vec{\mathrm{k}},\gamma} = \gamma\omega^\beta_{\vec{\mathrm{k}}}, \tag{8}$$

where $\gamma = \pm$. We define a general solution $\alpha$ as

$$\alpha_{\vec{k}} = u_{\vec{k},\uparrow} a_{\vec{k},\uparrow} + v_{\vec{k},\uparrow} b_{\vec{k},\uparrow} + u_{\vec{k},\downarrow} a^\dagger_{-\vec{k},\downarrow} + v_{\vec{k},\downarrow} b^\dagger_{-\vec{k},\downarrow}. \tag{9}$$

Next we have to solve the Bogoliubov - de Gennes (BdG) equations, which follow from the following condition,

$$[\alpha_{\vec{k}}, H_{BCS}] = E\alpha_{\vec{k}}. \tag{10}$$

From this matrix eigenvalue problem we obtain energies of the Bogoliubov quasiparticles,

$$\omega^p_{\vec{k}} = \pm\sqrt{(v_F|S(\vec{k})| + p\mu)^2 + |\Delta S(\vec{k})|^2}, \tag{11}$$

where $\pm$ stands for the particle and hole branches respectively for two kinds of excitations $p = -1(\alpha)$ and $p = +1(\beta)$. For $\mu = 0$ the system is gapless and we need a coupling $g$ larger than a critical value for the superconducting instability to exist [3]. This can be found considering in the BCS formalism the consistency or gap equation.

For each valley we have to solve the Bogoliubov problem using the expansion $S(\vec{K}_\pm + \vec{k}) \approx \mp a\sqrt{3}/2(k_x \mp ik_y)$. Near $K_+$ we need to diagonalize the following matrix, $M^*_{\vec{k}}$, that comes out from Eq. (10):

$$\begin{bmatrix} -\mu & v_F k & 0 & sk \\ v_F k^* & -\mu & sk^* & 0 \\ 0 & sk & \mu & -v_F k \\ sk^* & 0 & -v_F k^* & \mu \end{bmatrix},$$

where $s = s^* = -\Delta a\sqrt{3}/2 > 0$. Its eigenvectors (after normalization) enter the following expressions for Bogoliubov quasiparticles:

$$\begin{aligned}\alpha_{\vec{k},+} &= \frac{1}{2\sqrt{E_\alpha[E_\alpha-(\mu-v_F|k|)]}}\{[E_\alpha-(\mu-v_Fk)](\sqrt{\frac{k}{k^*}}a_{+\uparrow}+b_{+\uparrow})+\\ &+ s|k|(\sqrt{\frac{k}{k^*}}a^\dagger_{-\downarrow}+b^\dagger_{-\downarrow})\},\end{aligned} \tag{12}$$

and

$$\begin{aligned}\beta_{\vec{k},+} &= \frac{1}{2\sqrt{E_\beta[E_\beta-(\mu+v_F|k|)]}}\{[E_\beta-(\mu+v_Fk)](\sqrt{\frac{k}{k^*}}a_{+\uparrow}-b_{+\uparrow})-\\ &- s|k|(\sqrt{\frac{k}{k^*}}a^\dagger_{-\downarrow}-b^\dagger_{-\downarrow})\},\end{aligned} \tag{13}$$

and quasiholes:

$$\begin{aligned}\alpha_{\vec{k},-} &= \frac{1}{2\sqrt{E_\alpha[E_\alpha+(\mu-v_F|k|)]}}\{-[E_\alpha+(\mu-v_Fk)](\sqrt{\frac{k}{k^*}}a_{+\uparrow}+b_{+\uparrow})+\\ &+ s|k|(\sqrt{\frac{k}{k^*}}a^\dagger_{-\downarrow}+b^\dagger_{-\downarrow})\},\end{aligned} \tag{14}$$

and

$$\begin{aligned}\beta_{\vec{k},-} &= \frac{1}{2\sqrt{E_\beta[E_\beta+(\mu+v_F|k|)]}}\{-[E_\beta+(\mu+v_Fk)](\sqrt{\frac{k}{k^*}}a_{+\uparrow}-b_{+\uparrow})-\\ &- s|k|(\sqrt{\frac{k}{k^*}}a^\dagger_{-\downarrow}-b^\dagger_{-\downarrow})\},\end{aligned} \tag{15}$$

for the Bogoliubov solution near point $\vec{K}_+$, where we denoted $a_{\vec{K}_\pm\pm\vec{k},\sigma} \equiv a_{\pm\sigma}$ and $b_{\vec{K}_\pm\pm\vec{k},\sigma} \equiv b_{\pm\sigma}$.

The natural eigenstates of chirality appeared in our expressions. For example, $(\sqrt{\frac{k}{k^*}}a_{+\uparrow}+b_{+\uparrow})$ represents the spinor:

$$\chi = \begin{bmatrix} \sqrt{\frac{k^*}{k}} \\ 1 \end{bmatrix}, \tag{16}$$

which is the eigenstate of the chirality operator $\frac{\vec{\sigma}\vec{k}}{|k|}$, defined with $\vec{\sigma} = (\sigma_x, \sigma_y)$ Pauli matrices, i.e. the pseudospin (due to two sublattices) is along the momentum vector. The state $(\sqrt{\frac{k^*}{k}}a_{-\downarrow}+b_{-\downarrow})$ represents the same spinor because of the interchanged roles of sublattices at the $\vec{K}_-$ point. To see this in more detail we would like to remind the reader that instead of the Dirac free electron representation by the spinor

$$\chi^\dagger_{\vec{k}} = (a^\dagger_{\vec{K}_+ + \vec{k},\sigma}, b^\dagger_{\vec{K}_+ + \vec{k},\sigma} b^\dagger_{\vec{K}_- + \vec{k},\sigma}, a^\dagger_{\vec{K}_- + \vec{k},\sigma}), \tag{17}$$

and the chirality operator defined as

$$\begin{bmatrix} \frac{\vec{\sigma}\vec{k}}{|\vec{k}|} & 0 \\ 0 & -\frac{\vec{\sigma}\vec{k}}{|\vec{k}|} \end{bmatrix}, \tag{18}$$

in the BdG formalism we work with

$$\begin{aligned} \phi^\dagger_{\vec{k}} &= (a^\dagger_{\vec{K}_+ + \vec{k}\uparrow}, b^\dagger_{\vec{K}_+ + \vec{k}\uparrow} a_{\vec{K}_- - \vec{k}\downarrow}, b_{\vec{K}_- - \vec{k}\downarrow}) \\ &\equiv (a^\dagger_{+\uparrow}, b^\dagger_{+\uparrow} a_{-\downarrow}, b_{-\downarrow}). \end{aligned} \tag{19}$$

Note the reversed order of sublattices and the change of the sign of the momentum $\vec{k}$ near $\vec{K}_-$ point in the BdG formalism with respect to the free one. Thus, the lower $2 \times 2$ matrix on the diagonal of the Hamiltonian matrix in the free Dirac case can be read off from:

$$\begin{bmatrix} b^\dagger_{\vec{K}_- + \vec{k},\sigma} & a^\dagger_{\vec{K}_- + \vec{k},\sigma} \end{bmatrix} \begin{bmatrix} -\mu & -v_F k^* \\ -v_F k & -\mu \end{bmatrix} \begin{bmatrix} b_{\vec{K}_- + \vec{k},\sigma} \\ a_{\vec{K}_- + \vec{k},\sigma} \end{bmatrix}, \tag{20}$$

i.e. it is equal to $-v_F \vec{k}\vec{\sigma} - \mu$. Note that if we change the sign of $\vec{k}$ vector in Eq.(20) i.e. $\vec{k} \to -\vec{k}$ the off-diagonal elements in the matrix will change the sign, so that in this basis in the free representation the chirality operator will not have minus sign in the lower right entry of the matrix representation in Eq.(18). Therefore, $(\sqrt{\frac{k^*}{k}} a_{-\downarrow} + b_{-\downarrow})$ represents the same spinor (up to a phase factor) as in Eq.(16) and the same chirality eigenstate (with positive eigenvalue) as we pointed out earlier. Nevertheless, in the Bogoliubov representation we still have

$$\begin{bmatrix} a_{-\downarrow} & b_{-\downarrow} \end{bmatrix} \begin{bmatrix} \mu & -v_F k^* \\ -v_F k & \mu \end{bmatrix} \begin{bmatrix} a^+_{-\downarrow} \\ b^+_{-\downarrow} \end{bmatrix}, \tag{21}$$

i.e. the matrix is $-v_F \vec{k}\vec{\sigma} + \mu$, and the representation of the chirality operator remains the same as in Eq.(18). We will use this fact later on. On the other hand, the combinations in Eqs. (13) and (15): $(\sqrt{\frac{k}{k^*}} a_{+\uparrow} - b_{+\uparrow})$ and $(\sqrt{\frac{k^*}{k}} a_{-\downarrow} - b_{-\downarrow})$ have the pseudospin vector in the opposite direction of the momentum vector $\vec{k}$.

It is thus natural to introduce the following notation:

$$\sqrt{\frac{k}{k^*}} a_{+\uparrow} + b_{+\uparrow} \equiv c_{+\uparrow v}, \tag{22}$$

$$\sqrt{\frac{k}{k^*}} a^\dagger_{-\downarrow} + b^\dagger_{-\downarrow} \equiv c^\dagger_{-\downarrow v}, \tag{23}$$

$$\sqrt{\frac{k}{k^*}} a_{+\uparrow} - b_{+\uparrow} \equiv c_{+\uparrow w}, \tag{24}$$

$$-\sqrt{\frac{k}{k^*}}a^\dagger_{-\downarrow} + b^\dagger_{-\downarrow} \quad \equiv \quad c^\dagger_{-\downarrow w}, \tag{25}$$

where $v$ and $w$ denote the chirality, i.e. whether the pseudospin vector is along or in the opposite direction with respect to the $\vec{k}$ vector, respectively. We have to note that these electron operators are defined up to a phase factor, most importantly, the $\sqrt{\frac{k}{k^*}}$ phase. This degree of freedom should not influence the physics, but we chose the definitions so that the symmetry under exchange of particles in the ground state wave function is transparent.

The $\alpha$ and $\beta$ sectors are obviously decoupled in the Bogoliubov description, and we can concentrate on and examine closely the $\alpha$ sector first. Furthermore, we do not have to consider $\vec{K}_-$ point separately as the symmetry considerations tell us that the BdG equations around this point will induce the coupling or states of an electron around $\vec{K}_+$ point with $\downarrow$ projection of spin and those around $\vec{K}_-$ point with $\uparrow$ projection of spin.

Thus it suffices to consider the $\alpha$ sector first (with $c_{+\uparrow v}$ and $c_{-\downarrow v}$) and then use the symmetry arguments, more precisely antisymmetry under real spin exchange, to recover the whole ground state wave function. We can rewrite $\alpha$'s in the following form,

$$\alpha_{k,+} \quad = \quad u^p_k c_{+\uparrow v} + v^p_k c^\dagger_{-\downarrow v}, \tag{26}$$

$$\alpha_{k,-} \quad = \quad u^h_k c_{+\uparrow v} + v^h_k c^\dagger_{-\downarrow v}. \tag{27}$$

We should demand $\alpha_{k,+}|G\rangle = 0$ and $\alpha^\dagger_{k,-}|G\rangle = 0$, for any $k$, if $|G\rangle$ is to represent the ground state vector. That implies that in the $\alpha$ sector of $\vec{K}_+$ point we have the following contribution to the ground state,

$$\prod_k (u^p_k - v^p_k c^\dagger_{+\uparrow v} c^\dagger_{-\downarrow v})|0\rangle, \tag{28}$$

where $|0\rangle$ denotes the vacuum. This state is annihilated with both, $\alpha_{k,+}$ and $\alpha^\dagger_{k,-}$. The symmetry arguments demand that we should get a similar expression considering the BdG equations at $\vec{K}_-$ point. If we denote by $g_\alpha(k) = -\frac{v^p_k}{u^p_k}$, the ground state vector in the $\alpha$ sector should look like:

$$\begin{aligned} & \prod_k (1 + g_\alpha(k) c^\dagger_{+\uparrow v} c^\dagger_{-\downarrow v})(1 + g_\alpha(k) c^\dagger_{-\uparrow v} c^\dagger_{+\downarrow v})|0\rangle = \\ = \; & \prod_k \{1 + g_\alpha(k)[c^\dagger_{+\uparrow v} c^\dagger_{-\downarrow v} + c^\dagger_{-\uparrow v} c^\dagger_{+\downarrow v}] + \frac{g^2_\alpha(k)}{2}[c^\dagger_{+\uparrow v} c^\dagger_{-\downarrow v} + c^\dagger_{-\uparrow v} c^\dagger_{+\downarrow v}]^2|0\rangle \\ = \; & \exp\{\sum_k g_\alpha(k)[c^\dagger_{+\uparrow v} c^\dagger_{-\downarrow v} + c^\dagger_{-\uparrow v} c^\dagger_{+\downarrow v}]\}|0\rangle. \end{aligned} \tag{29}$$

Now we can identify $g_\alpha(k)$ to represent a Fourier transform of the wave function of a Cooper pair of electrons, which is a spin-singlet with respect to spin degree of freedom and a triplet state (symmetric under exchange) with respect to the valley ($K_\pm$) degree of freedom. If we defined differently our electron operators, there would be the possibility for $g_\alpha(k)$ to acquire the phase factor $\sqrt{\frac{k}{k^*}}$, which would make the identification of the antisymmetry under exchange harder.

Taking into account the $\beta$ sector (with the chirality in the opposite direction of the momentum: $w$) the complete ground state vector is

$$\exp\{\sum_k g_\alpha(k)[c^\dagger_{+\uparrow v}c^\dagger_{-\downarrow v} + c^\dagger_{-\uparrow v}c^\dagger_{+\downarrow v}] + \sum_k g_\beta(k)[c^\dagger_{+\uparrow w}c^\dagger_{-\downarrow w} + c^\dagger_{-\uparrow w}c^\dagger_{+\downarrow w}]\}|0\rangle, \tag{30}$$

where

$$g_\alpha(k) = -\frac{s|k|}{E_\alpha - (\mu - v_F|k|)} \quad \text{and} \quad \mathrm{g_\beta(k) = -\frac{s|k|}{E_\alpha - (\mu + v_F|k|)}}. \tag{31}$$

Using the long-distance (low-momentum) expansions for $E_\alpha$ and $E_\beta$, for a finite $\mu$,

$$E_{\alpha(\beta)} \approx \mu \mp v_F|k| + \frac{s^2|k|^2}{2\mu}, \tag{32}$$

we find the long-distance behavior of the pair wave function to be

$$\lim_{|\vec{r}|\to\infty} g_\alpha(\vec{r}) = \lim_{|\vec{r}|\to\infty} g_\beta(\vec{r}) \sim \frac{1}{|\vec{r}|}, \tag{33}$$

i.e. we have a case for a weak coupling [7]. As emphasized in Ref. 7, the term weak pairing does not mean also weak coupling; it stands for a phase with an unusual large spread of the Cooper pairs. On the other hand, for $\mu = 0$ we have that $g_\alpha(k)$ and $g_\beta(k)$ are two constants and the Cooper pairs are localized on a short scale $\sim a$ in the graphene system at the neutrality point. Thus for $\mu = 0$ we have a case for a strong pairing.

The ground state vector (wave function) in Eq.(30) displays two kinds of Cooper pairs, each antisymmetric under combined exchange of (a) orbital, (b) valley ($\vec{K}_\pm$), and (c) spin ($\uparrow, \downarrow$) degree of freedom. Two kinds of Cooper pairs stem from the chirality (sublattice) degree of freedom intimately connected with the Dirac-nature of the electron with both, particles and holes. They both, particles (with positive chirality $v$ at $\vec{K}_+$) and holes (with negative chirality $w$ at $\vec{K}_+$), constitute Cooper pairs, which are symmetric under $v \leftrightarrow w, v_F \to -v_F$ transformation.

In the long distance limit we recover the form of the wave function of ordinary $s$-wave superconductor as given in Ref. 9, though with more, two-component, degrees of freedom. The Cooper pair wave function is antisymmetric under spin exchange and symmetric under exchange of valley ($\vec{K}_\pm$), sublattice ($v, w$), and orbital degrees of freedom.

Next we will discuss the spin-triplet case, more precisely we will assume that the system is spin-polarized and not consider spin in the following. Therefore, fermions are spinless just like in the Pfaffian case, but they live on the honeycomb lattice. We will assume $\langle a_i b_j\rangle = \Delta$. In this case the Bogoliubov problem in Eq.(5) for the spin-singlet pairing transforms into a similar one with $a_{\vec{k},\sigma} \equiv a_{\vec{k}}$ and $b_{\vec{k},\sigma} \equiv b_{\vec{k}}$, and the matrix $M_{\vec{k}}$ becomes as follows

$$M_{\vec{k}} = \begin{bmatrix} -\mu & -tS(\vec{k}) & 0 & \Delta S(\vec{k}) \\ -tS^*(\vec{k}) & -\mu & -\Delta S(-\vec{k}) & 0 \\ 0 & -\Delta S^*(-\vec{k}) & \mu & tS(\vec{k}) \\ \Delta S^*(\vec{k}) & 0 & tS^*(\vec{k}) & \mu \end{bmatrix}.$$

Around the $\vec{K}_+$ point we have

$$\begin{bmatrix} -\mu & v_F k^* & 0 & sk^* \\ v_F k & -\mu & -sk & 0 \\ 0 & -sk^* & \mu & -v_F k^* \\ sk & 0 & -v_F k & \mu \end{bmatrix},$$

where $s = -\Delta a \frac{\sqrt{3}}{2} > 0$ as before. The problem around the $\vec{K}_-$ point is a copy of the problem around the $\vec{K}_+$ point.

Now the $M_{\vec{k}}$ matrix around $\vec{K}_+$ point cannot be cast, as in the spin-singlet case, in the following form,

$$\begin{bmatrix} v_F \vec{\sigma}\vec{k} - \mu I_2 & s\vec{\sigma}\vec{k} \\ s\vec{\sigma}\vec{k} & -v_F \vec{\sigma}\vec{k} + \mu I_2 \end{bmatrix},$$

where $I_2$ is the $2 \times 2$ identity matrix, which commutes with the chirality matrix (Eq.18). $M_{\vec{k}}$ around $\vec{K}_+$ point can be compactly written as

$$\begin{bmatrix} v_F \vec{\sigma}\vec{k} - \mu I_2 & si\vec{k} \times \vec{\sigma} \\ -si\vec{k} \times \vec{\sigma} & -v_F \vec{\sigma}\vec{k} + \mu I_2 \end{bmatrix},$$

and it does not commute with the chirality operator. The eigenstates of the Bogoliubov problem do not have to be the eigenstates of chirality. We find the following eigenvalues $E_p = \pm\sqrt{\mu^2 + |\vec{k}|^2 s^2 + |\vec{k}|^2 v_F^2 + p\, 2\sqrt{\mu^2 v_F^2 |\vec{k}|^2 + s^2 v_F^2 |\vec{k}|^2}}$, where $p = +1(\alpha)$ and $p = -1(\beta)$ are two branches as before. The associated eigenvectors can be written as sums of fermionic particle eigenstates of chirality only at the low-momentum limit and we list those connected with positive eigenvalues,

$$\alpha_{\vec{k},+} = \frac{1}{\sqrt{2(1 + \frac{|k|^2 s^2}{4\mu^2})}}[\frac{|k|s}{2\mu}(-\sqrt{\frac{k}{k^*}}a_+ + b_+) + (\sqrt{\frac{k}{k^*}}a_-^\dagger + b_-^\dagger)], \tag{34}$$

and

$$\beta_{\vec{k},+} = \frac{1}{\sqrt{2(1 + \frac{|k|^2 s^2}{4\mu^2})}}[-\frac{|k|s}{2\mu}(\sqrt{\frac{k}{k^*}}a_+ + b_+) + (-\sqrt{\frac{k}{k^*}}a_-^\dagger + b_-^\dagger)], \tag{35}$$

and negative eigenvalues,

$$\alpha_{\vec{k},-} = \frac{1}{\sqrt{2(1 + \frac{4\mu^2}{|k|^2 s^2})}}[\frac{2\mu}{|k|s}(\sqrt{\frac{k}{k^*}}a_+ + b_+) + (-\sqrt{\frac{k}{k^*}}a_-^\dagger + b_-^\dagger)], \tag{36}$$

and

$$\beta_{\vec{k},-} = \frac{1}{\sqrt{2(1 + \frac{4\mu^2}{|k|^2 s^2})}}[\frac{2\mu}{|k|s}(\sqrt{\frac{k}{k^*}}a_+ - b_+) + (\sqrt{\frac{k}{k^*}}a_-^\dagger + b_-^\dagger)]. \tag{37}$$

Similarly as before we can define

$$\sqrt{\frac{k}{k^*}}a_+ + b_+ \equiv c_{+v}, \tag{38}$$

$$\sqrt{\frac{k}{k^*}}a_-^\dagger + b_-^\dagger \equiv c_{-v}^\dagger, \tag{39}$$

$$\sqrt{\frac{k}{k^*}}a_+ - b_+ \equiv c_{+w}, \tag{40}$$

$$-\sqrt{\frac{k}{k^*}}a_-^\dagger + b_-^\dagger \equiv c_{-w}^\dagger, \tag{41}$$

and the ground state vector can be cast in the following form,

$$\exp\{\sum_{\vec{k}} \frac{2\mu}{s|k|}(c_{+,v}^\dagger c_{-,w}^\dagger + c_{+,w}^\dagger c_{-,v}^\dagger)\}|0\rangle. \tag{42}$$

In this case, each Cooper pair is antisymmetric under exchange of $\vec{K}_\pm$ points, i.e. valley degree of freedom and symmetric under exchange of sublattices, i.e. chirality ($v \leftrightarrow w$). Depending on our definitions for $c$'s two degrees of freedom can exchange the symmetry properties. We find again the weak pairing ($\sim \frac{1}{r}$) behavior in the orbital part.

## 6. Conclusion

We derived the ground state wave functions for the superconductivity on the honeycomb lattice induced by nearest-neighbor attractive interactions and with order parameter independent of the direction to any of the nearest neighbors. Although the order parameter in momentum space has the $p \pm ip$ form in a low effective description the Cooper pair wave function behaves as an $s$-wave (with no angular dependence) and decays as $\sim \frac{1}{r}$. Other (discrete) degrees of freedom combine to make the Cooper pair antisymmetric under exchange. At the point of the transition, $\mu = 0$, in the spin-singlet case, a strong pairing (of the order of lattice spacing) occurs.

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